

A remark on kinks and time machines

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Abstract

We describe an elementary proof that a manifold with the topology of the Pöltner time machine does not admit a nonsingular, asymptotically flat Lorentz metric.

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in a recent paper Chamblin, Gibbons and Steif [1] have used the idea of gravitational kinks [2-3] to prove that in even spacetime dimensions there does not exist a nonsingular, asymptotically flat Lorentz metric on the smooth Politzer time machine. The $7L$ -dimensional Politzer time machine is obtained by (i) starting from the Minkowski spacetime \mathbb{R}^n with the flat metric, (ii) by cutting out and duplicating the two closed unit $(n-1)$ -balls centered at the origin in each of the spacelike hypersurfaces $\{t=0\}$ and $\{t=1\}$, so that a pair of spherical "slits" is created, one in each hypersurface, and (iii) by identifying the upper edge of the first slit with the lower edge of the second, and the lower edge of the first slit with the upper edge of the second. The resulting topological space is not a manifold, as the boundary points of the two slits (the points that lie on the boundary spheres of the two unit $(n-1)$ -balls) do not have locally Euclidean open neighborhoods. However, these singularities can be trivially smoothed-out to obtain a manifold with the same global topology as the original space; we will call this manifold "smooth Politzer space" for emphasis. It has the topology of a n -dimensional "handle" i.e. $S^{(n-1)} \times S^1$ with a point (which corresponds to infinity) removed. (A more drastic way to obtain a smooth manifold is to simply remove the troublesome boundary points of the two slits; together with the flat metric on the original space, this yields a well-defined spacetime with singularities where the boundary points are removed. It is this space that is usually referred to as the "Politzer spacetime" in the literature.) The question addressed by [1], and also by the present note, is whether a smooth Lorentz metric, which has the same qualitative behavior as the flat metric on the original Politzer space, exists on the smooth Politzer manifold. In particular, such a metric needs to be asymptotically flat at the asymptotically Euclidean "end" (the point at "infinity" removed from $S^{(n-1)} \times S^1$) of the Politzer handle. (For more detailed background information we refer to [1].)

This question has been answered negatively in [1] by Chamblin, Gibbons and Steif. The techniques used by these authors, based on the general notion of gravitational kinks [2-3], have wide scope and great power in dealing with questions of this kind. However, for the present specific question about the smooth Politzer space, there exists a simpler proof of the negative answer that uses only elementary differential topology; we will now present this proof. Our argument is based on a single well-known result: the Poincaré-Hopf theorem [4], which states that for a compact, orientable manifold M and a smooth vector field X on M with only isolated zeros, the Euler number $\chi(M)$ is equal to the sum over the zeros of X of the indices of X at those zeros. One immediate corollary of this result is that if $\chi(M) \neq 0$, a compact, orientable M does not

admit an everywhere non-zero vector field, and hence do not admit a smooth Lorentz metric. Another corollary is the formula for the Euler number of a connected sum: If M_1 and M_2 are compact, orientable even $(2m-)$ dimensional manifolds, then $\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - 2$. To derive this from the Poincaré-Hopf, consider vector fields X_1 and X_2 with isolated zeros on M_1 and M_2 , respectively, so that X_i has a simple zero at $p_i \in M_i$. Since the antipodal map on the odd-sphere $S^{(2m-1)}$ is homotopic to identity ([4]), we can, without changing the indices at the zeros, flip the signs of the X_i if necessary and arrange that X_1 is inward-pointing near p_1 and X_2 is outward-pointing near p_2 . Now we perform the connected sum by joining M_1 and M_2 in a neighborhood of $p_i \in M_i$. Then we can smoothly extend the vector fields X_i to a vector field X on $M_1 \# M_2$ such that X has precisely the same set of zeros and indices as the X_i except for the simple zeros at p_1 and p_2 : X is smooth and non-zero near $p_1 \equiv p_2$ by construction. The formula now follows by simple counting.

Now assume that the smooth Politzer space $S^1 \times S^{(n-1)} \setminus p_\infty$ admits a smooth Lorentz metric which is asymptotically flat near p_∞ . Let T^n denote the n -torus with the standard flat Lorentz metric on it. Since the Lorentz metric on the Politzer manifold is asymptotically flat at p_∞ , we can combine the two metrics smoothly to obtain a Lorentz metric on the connected sum $(S^1 \times S^{(n-1)}) \# T^n$. But the Euler number of $S^1 \times S^{(n-1)}$ and of T^n are both zero. If the dimension n is even, by the formula above $\chi[(S^1 \times S^{(n-1)}) \# T^n] = -2$, accordingly, it is impossible for $(S^1 \times S^{(n-1)}) \# T^n$ to admit a smooth Lorentz metric. This contradiction proves the result we seek in even dimensions n .

More generally, let us call a n -manifold M "asymptotically Euclidean" if compact subsets $K \subset M$ and $B \subset \mathbb{R}^n$ exist such that $M \setminus K$ is diffeomorphic to $\mathbb{R}^n \setminus B$. Call a Lorentz metric on M asymptotically flat if on $M \setminus K$ it is asymptotic to the Minkowski metric. For an asymptotically Euclidean M , let M^∞ denote the one-point compactification of M obtained by smoothly adjoining to M the point at infinity of $M \setminus K$. Thus, if M is \mathbb{R}^n , then M^∞ is S^n ; for the smooth Politzer space M^∞ is $S^1 \times S^{(n-1)}$. The argument above proves the following result:

Let M be an orientable, even-dimensional asymptotically Euclidean manifold. If $\chi(M^\infty) \neq 2$, then M does not admit a smooth, asymptotically flat Lorentz metric.

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